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Distributions of absolute central moments for random walk surfaces

A J McKane† and R K P Zia‡

† Department of Theoretical Physics, University of Manchester, Manchester M13 9PL, UK

‡ Center for Stochastic Processes in Science and Engineering and Department of Physics, Virginia Polytechnic and State University, Blacksburg, Virginia 24061, USA

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Abstract. We study periodic Brownian paths, wrapped around the surface of a cylinder. One characteristic of such a path is its width square, w^2, defined as its variance. Though the average of w^2 over all possible paths is well known, its full distribution function was investigated only recently. Generalizing w^2 to $w^{(N)}$, defined as the Nth power of the *magnitude* of the deviations of the path from its mean, we show that the distribution functions of these also scale and obtain the asymptotic behaviour for both large and small $w^{(N)}$.

1. Introduction

Of all the quantities which characterize a near-planar surface, the width, w, defined as the standard deviation of the position of the surface, is probably the most frequently studied. If this surface is a member of a Gaussian ensemble, then $\langle w^2 \rangle$, the ensemble average of w^2, is a well known function of the parameters of the Gaussian as well as the size of the surface. In particular, if the surface is a one-dimensional object of length T, then $\langle w^2 \rangle \propto T$, as $T \to \infty$. In a recent paper [1] it was pointed out that the entire *distribution* of w^2, rather than merely the average, would provide a better picture of the surface. Indeed, this distribution was shown to obey scaling and the universal scaling function was computed. An analogue in critical phenomena is the difference between measuring a single critical exponent and finding data collapse in a range of parameter space. It is indisputable that the latter gives far more details of the system than the former.

In a similar vein, one may extract further information from the fluctuating surfaces by analysing moments or cumulants other than the second. For Gaussian surfaces, the *averages* of these can be related to $\langle w^2 \rangle$, but their *distributions* are less trivial. Unlike the case for w^2, the Laplace transforms of these distributions are associated with non-Gaussian field theories. Equivalently, for one-dimensional surfaces, we must now deal with quantum anharmonic oscillators. In this paper we generalize the analysis begun in [1] and show that these distributions also scale, while their universal functions can be found in the asymptotic regimes where the argument is small or is large.

In order to have sufficient analytical control over the fluctuations, the one-dimensional surfaces in [1] were modelled by Brownian paths $\{h_t\}$ in the 'time' interval $0 \leqslant t \leqslant T$, with periodic boundary conditions. Thus, a surface may be thought of as a line wrapped around a cylinder of circumference T. By identifying the continuous path h_t as the height of a surface, the 'width square' is a simple functional quadratic in h_t:

$$w^2[h_t] = \overline{\psi_t^2} = \overline{\left[h_t - \overline{h_t}\right]^2} \tag{1}$$

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where the average, $\overline{f}$, of a function f is defined as

$$\overline{f(h_t)} = \frac{1}{T}\int_0^T \mathrm{d}t\, f(h_t) \tag{2}$$

and where $\psi_t \equiv h_t - \overline{h_t}$ is the usual deviation from the 'average height'. Being Brownian paths, the weight of each h_t is a simple Gaussian, i.e. $\exp\{-\int_0^T \mathrm{d}t\,[\frac{1}{2}\dot{h}_t^2]\}$, where $\dot{h}_t = \mathrm{d}h_t/\mathrm{d}t$. For the ensemble average, denoted by $\langle \ldots \rangle$, one must sum over all appropriate paths, i.e. perform functional integrals (or path-integrals [2]) over periodic functions h_t. Since none of the quantities we encounter will depend on the overall height, we may just as well sum over $\{\psi_t\}$, with special attention paid to the constraint $\int_0^T \mathrm{d}t\,\psi_t = 0$. With this set-up, it is standard practice to show that $\langle w^2 \rangle = T/12$.

The key to calculating the probability density $P(w^2; T)$ was to express it the form of a path-integral and note that its Laplace transform can be computed explicitly [1]. Beginning with

$$\begin{aligned} P(w^2;T) &= \left\langle \delta\left(w^2 - \overline{\left[h_t - \overline{h_t}\right]^2}\right)\right\rangle \\ &= \mathcal{N}\int \mathcal{D}[h]\delta\left(w^2 - \overline{\left[h_t - \overline{h_t}\right]^2}\right)\exp\left(-T\overline{\dot{h}_t^2}/2\right) \end{aligned} \tag{3}$$

where $\mathcal{N}$ is a normalization constant, one can easily show that it has the scaling form $P(w^2;T) = \langle w^2\rangle^{-1}\Phi(x)$, where x is the scaling variable $w^2/\langle w^2\rangle$. Inverting the exact result for its Laplace transform

$$G(\lambda;T) = \frac{\sqrt{\lambda T/2}}{\sinh(\sqrt{\lambda T/2})} \tag{4}$$

the associated scaling function is obtained:

$$\Phi(x) = \frac{\pi^2}{3}\sum_{n=1}^{\infty}(-1)^{n-1}n^2\exp\left(-\frac{\pi^2}{6}n^2 x\right) \tag{5}$$

from which the asymptotic properties can be found.

The aim of this paper is to extend these results to arbitrary absolute central moments:

$$w^{(N)}[h_t] \equiv \overline{|\psi_t|^N} = \overline{|h_t - \overline{h_t}|^N}. \tag{6}$$

Notice that, for even N, these are precisely the central moments associated with a particular configuration h_t. Using the absolute values of ψ, it is possible to study arbitrary $N > 0$, and these are known as the absolute central moments (associated with a specific path). The outline of the paper is as follows. In section 2, a brief, self-contained, summary of the formalism for this general case is included. Then, we show that $P_N(w^{(N)};T)$ does indeed scale. Analytic expressions for the asymptotic behaviour of the scaling function are found, for large and small arguments, in sections 3 and 4 respectively. We conclude with some general comments in section 5.

2. Scaling form for the probability distribution

Following [1], let us express the full distribution of $w^{(N)}$ as

$$P_N(w^{(N)};T) = \mathcal{N}\int \mathcal{D}[h]\delta\left(w^{(N)} - \overline{|h_t - \overline{h_t}|^N}\right)\exp\left(-T\overline{\dot{h}_t^2}/2\right). \tag{7}$$

Its Laplace transform, which generates the moments of (7), is

$$G_N(\lambda; T) = \int_0^\infty \mathrm{d}\zeta\ P_N(\zeta; T)\mathrm{e}^{-\lambda\zeta} = \mathcal{N} \int \mathcal{D}[h] \exp\left[-T\overline{\dot{h}_t^2}/2 - \lambda\overline{|h_t - \overline{h_t}|^N}\right]. \tag{8}$$

Writing this expression in terms of ψ_t and using the condition $G_N(0; T) = 1$ to determine the normalization constant $\mathcal{N}$, we obtain

$$G_N(\lambda; T) = \frac{\int \mathcal{D}[\psi]\delta(\int_0^T \mathrm{d}t\ \psi_t) \exp - \int_0^T \mathrm{d}t\ [\frac{1}{2}\dot{\psi}_t^2 + \frac{\lambda}{T}|\psi_t|^N]}{\int \mathcal{D}[\psi]\delta(\int_0^T \mathrm{d}t\ \psi_t) \exp - \int_0^T \mathrm{d}t\ [\frac{1}{2}\dot{\psi}_t^2]}. \tag{9}$$

Apart from the constraint $\int_0^T \mathrm{d}t\ \psi_t = 0$, this is the imaginary time propagator for a quantum-mechanical particle moving in the potential $|\psi|^N$. As we will see in section 4, this connection will be exploited in the study of the asymptotics of P_N for small $w^{(N)}$.

To elucidate the general structure of P_N and its scaling properties, it is useful to define a dimensionless variable $\tau = t/T$ and then to make the following rescaling:

$$\chi_\tau = T^{-1/2}\psi_\tau. \tag{10}$$

Defining

$$\mu = N\lambda T^{N/2} \tag{11}$$

we have, for the generating function,

$$G_N(\lambda; T) = \frac{\int \mathcal{D}[\chi]\delta(\int_0^1 \mathrm{d}\tau\ \chi_\tau) \exp - \int_0^1 \mathrm{d}\tau\ [\frac{1}{2}\dot{\chi}_\tau^2 + (\mu/N)|\chi_\tau|^N]}{\int \mathcal{D}[\chi]\delta(\int_0^1 \mathrm{d}\tau\ \chi_\tau) \exp - \int_0^1 \mathrm{d}\tau\ [\frac{1}{2}\dot{\chi}_\tau^2]} \tag{12}$$

where $\dot{\chi}_\tau$ now means $\mathrm{d}\chi_\tau/\mathrm{d}\tau$. This expression shows that $G_N(\lambda; T)$ is a function of μ only, so that we may write

$$g_N(\mu) \equiv G_N(\lambda; T). \tag{13}$$

Expressing $P_N(w^{(N)}; T)$ as the inverse Laplace transform of g:

$$P_N(w^{(N)}; T) = \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}} g_N(N\lambda T^{N/2}) \exp(\lambda w^{(N)}) = \frac{1}{NT^{N/2}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{d}\mu}{2\pi\mathrm{i}} g_N(\mu) \exp\left(\frac{\mu}{N}\frac{w^{(N)}}{T^{N/2}}\right) \tag{14}$$

we see that $T^{N/2}P_N(w^{(N)}; T)$ is a function of $w^{(N)}/T^{N/2}$ only. To put this in the scaling form, we simply note that $\langle w^{(N)}\rangle$ is proportional to $T^{N/2}$:

$$\langle w^{(N)}\rangle = \int_0^\infty \mathrm{d}\zeta\ \zeta P_N(\zeta; T) = -\left.\frac{\mathrm{d}G_N}{\mathrm{d}\lambda}\right|_{\lambda=0} = -NT^{N/2}\left.\frac{\mathrm{d}}{\mathrm{d}\mu}g_N(\mu)\right|_{\mu=0}. \tag{15}$$

Using this result, (14) may be written as

$$P_N(w^{(N)}; T) = \frac{1}{\langle w^{(N)}\rangle}\Phi_N(x) \tag{16}$$

where

$$x \equiv w^{(N)} / \langle w^{(N)} \rangle \tag{17}$$

and

$$\Phi_N(x) = |g'_N(0)| \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{d}\mu}{2\pi\mathrm{i}} g_N(\mu) \exp(\mu x |g'_N(0)|). \tag{18}$$

Equations (16) and (17) are an expression of the fact that P_N scales. From (18) one sees that, in order to find an explicit form for $\Phi_N(x)$, the function $g_N(\mu)$ has to be determined. For $N \neq 2$ this can only be determined in special limits. Nevertheless, as we shall see in the next two sections, one can go quite a long way to finding the limiting forms of Φ_N for large and small x using path-integral techniques.

We end this section by noting that, since we have Brownian paths, the ensemble averages of the moments $\langle w^{(N)} \rangle$ are quite simple. For even N, they are $(N-1)!!(T/12)^{N/2}$. For arbitrary N, we rely on standard analytic continuation techniques to arrive at

$$|g'_N(0)| = \Gamma\left(\frac{N+1}{2}\right) \Big/ \{6^{N/2} N \pi^{1/2}\}. \tag{19}$$

However, we will continue to write $g'_N(0)$, both for convenience and to emphasize its role.

3. The scaling function for large x

It is not surprising that the behaviour of $\Phi_N(x)$ for $x \gg 1$ is controlled by the singularities of $g_N(\mu)$ for near the origin of μ. In the $N = 2$ case, these singularities are simple poles on the negative real axis. In general, we expect a branch cut and its discontinuity near $\mu = 0$ to dominate the asymptotics of Φ_N. One approach is to compute these singularities, using techniques of asymptotic expansion around instanton solutions [3], followed by an inversion of the Laplace transform. Alternatively, one can insert (12) into (18), find the saddle point in the combined space of $\{\mu, \chi_\tau\}$ and perform the Gaussian integrations over small variations in its neighbourhood. We shall follow the latter route, which seems to be simpler.

As will be seen, we are able to compute explicitly only the leading two terms in $\ln \Phi_N$, i.e. $x^{2/N}$ and $\ln x$. Thus, we will drop all proportionality constants, for clarity, when writing Φ_N. In particular, the denominator of (12) is clearly x independent and will be suppressed, though its role in regulating the Gaussian integrations is obviously essential. So, consider

$$\Phi_N(x) \propto \int \mathrm{d}\mu\, \mathcal{D}[\chi] \delta\left(\int_0^1 \mathrm{d}\tau\, \chi_\tau\right) \exp\left\{\mu x |g'_N(0)| - \int_0^1 \mathrm{d}\tau \left[\frac{1}{2}\dot{\chi}_\tau^2 + \frac{\mu}{N}|\chi_\tau|^N\right]\right\}. \tag{20}$$

The attentive reader may object to the factor i in (18) being dropped as well, since Φ_N should be real. A careful analysis will reveal that the functional determinant carries this factor, i.e. the contour for μ near the saddle point will be parallel to the imaginary axis.

From (20), we first seek the saddle point, to be denoted by $\{-M, X_\tau\}$ in $\{\mu, \chi_\tau\}$ space. Note that we have anticipated this point to be on the negative real μ axis, so that M will turn out to be real and positive. Setting to zero the variation of the exponent in (20) with respect to μ and χ_τ, we have the equations for the paths which dominate the path-integral:

$$N x |g'_N(0)| = \int_0^1 \mathrm{d}\tau\, |X_\tau|^N \tag{21}$$

and

$$\ddot{X}_\tau + M\, \mathrm{sgn}(X_\tau) |X_\tau|^{(N-1)} = 0. \tag{22}$$

Note that, in order to ignore the singularity at $X = 0$, we must restrict our attention to $N > 1$ here. The latter being an equation of motion for a classical particle of unit mass moving in a potential $V(X) = \frac{M}{N}|X|^N$, these can be solved easily. The importance of $M > 0$ is also now evident, since periodic χ_τ's would have been otherwise impossible. Finally, the constraint $\int_0^1 \mathrm{d}\tau\, \chi_\tau = 0$ can be satisfied trivially.

Integrating (22) once, we have

$$\frac{1}{2}\dot{X}_\tau^2 + \frac{M}{N}|X_\tau|^N = \epsilon \tag{23}$$

where ϵ is a constant, representing the total energy in the mechanical analogy. Assigning $\tau = 0$ to the point of maximum amplitude (and so zero velocity), we set

$$\epsilon = \frac{M}{N}X_0^N \tag{24}$$

where $X_0 > 0$. Equation (23) can be integrated again in the usual manner, exploiting both the periodic boundary conditions and the constraint. Clearly, we need to focus on only a quarter of the period, so that

$$\tfrac{1}{4} = X_0^{(2-N)/2}(N/2M)^{1/2}\int_0^1 \mathrm{d}\xi\, (1-\xi^N)^{-1/2} \tag{25}$$

where $\xi_\tau \equiv X_\tau/X_0$. The integral is proportional to the beta function $B(1/N, \frac{1}{2})$, which we will denote simply as B. Now we have a relation between M and X_0:

$$X_0^{(N-2)} = \frac{8B^2}{NM}. \tag{26}$$

Inserting this into equation (21), eliminating $\mathrm{d}\tau$ in favour of $\mathrm{d}X/\dot{X}$ and using (23), we have

$$Nx|g_N'(0)| = 4\int_0^{X_0}(\mathrm{d}X/\dot{X})|X|^N \tag{27}$$

$$= \left(\frac{8X_0^{N+2}}{MN}\right)^{1/2} B(1+1/N, \tfrac{1}{2}). \tag{28}$$

Using $B(1+\alpha,\beta) = B(\alpha,\beta)\alpha/(\alpha+\beta)$, we find both X_0 and M in terms of the scaling variable x:

$$X_0 = \{N(\tfrac{1}{2}N+1)x|g_N'(0)|\}^{1/N} \tag{29}$$

and

$$M = \left(\frac{8B^2}{N}\right)\{N(\tfrac{1}{2}N+1)x|g_N'(0)|\}^{(2-N)/N}. \tag{30}$$

With all parameters of the saddle point explicitly determined, we evaluate the exponential in (20), i.e. the total 'action'. The result is $-MNx|g_N'(0)|/2$, so that the leading asymptotic behaviour of Φ_N is

$$\Phi_N(x) \sim \exp\{-\mathcal{S}_N x^{2/N}\} \tag{31}$$

$$\mathcal{S}_N = 4B^2\{N(\tfrac{1}{2}N+1)\}^{(2-N)/N}|g_N'(0)|^{(2/N)}. \tag{32}$$

For $N = 2$, $B = \pi$ and, from (19), $|g_N'(0)| = \frac{1}{24}$, so that we recover the result in (5).

Next, we turn to the computation of the prefactor, for which it is necessary to study the Gaussian fluctuations about the saddle point $(-M, X_\tau)$.

To accomplish this, we write $\mu = -M + \hat{\mu}$ and $\chi_\tau = X_\tau + \hat{\chi}_\tau$ in (20) and shift the integration to $(\hat{\mu}, \hat{\chi}_\tau)$. Expecting the integrals to be dominated by small $(\hat{\mu}, \hat{\chi}_\tau)$, we expand the argument of the exponential to second order in these fluctuations. The zeroth order is given above in (31) while the first order vanishes by the choice of the saddle point. At the second order, the result is the quadratic form

$$-\hat{\mu}\int_0^1 \mathrm{d}\tau\ \mathrm{sgn}(X_\tau)|X_\tau|^{(N-1)}\hat{\chi}_\tau + \tfrac{1}{2}\int_0^1 \mathrm{d}\tau\ \hat{\chi}_\tau \mathcal{M}\hat{\chi}_\tau \tag{33}$$

where $\mathcal{M}$ is the operator

$$\mathcal{M} = -\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} - M(N-1)|X_\tau|^{(N-2)}. \tag{34}$$

Note that, strictly speaking, we should impose $N \geqslant 2$, so that the singularity at $X = 0$ can be ignored here also.

Before carrying out the integration over $(\hat{\mu}, \hat{\chi}_\tau)$, we discuss several important points.

First, note that $\mathcal{M}$ is a Hermitian, Schrödinger-like operator. It can be diagonalized, and, since our problem is based on a finite interval with periodic boundary conditions, it has a real, discrete spectrum with real, periodic eigenmodes. Thus, $\hat{\chi}$ can be expanded in terms of these modes and the functional integral $\int \mathcal{D}[\hat{\chi}]$ can be defined as over the amplitudes in this expansion. We will demonstrate that there is a single zero eigenvalue, associated with a zero mode, which must be handled with some care. Defining $\mathcal{M}'$ to be the operator (34) *restricted to the subspace orthogonal to this mode*, we will argue that the spectrum of $\mathcal{M}'$ is positive, so that its inverse is well defined and the Gaussian integration in (33) over these modes is simple. Furthermore, the product $M|X_\tau|^{(N-2)}$ in (34) can be written as $(MX_0^{(N-2)})|\xi_\tau|^{(N-2)}$. But, according to (26), this quantity is independent of x, so that the spectra of both $\mathcal{M}$ and $\mathcal{M}'$ are also x-independent. Thus, the result of the Gaussian integration, which involves $\det \mathcal{M}'$, is also x-independent. Since we are dropping all proportionality constants in this study, this factor will be neglected.

Second, we will show that the zero mode is also absent from the off-diagonal $\hat{\mu} - \hat{\chi}$ part. As a result, at the quadratic level, the $\hat{\chi}$ integration leads to a distribution for $\hat{\mu}$ of the form $\exp(\mathcal{P}\hat{\mu}^2/2)$, where

$$\mathcal{P} = \int \mathrm{d}\tau\, \mathrm{d}\tau'\, |X_\tau|^{(N-1)}\mathcal{M}'^{-1}|X_{\tau'}|^{(N-1)} \tag{35}$$

is positive. To make sense of performing such an integral (over $\hat{\mu}$), recall that the contour in the $\hat{\mu}$-plane is parallel to the imaginary axis, by definition of the inverse Laplace transform. We have simply chosen to have it run through a saddle point on the real axis: $\mu = -M$. Thus, it is entirely consistent to choose $\hat{\mu}$ to be pure imaginary, so that not only is the Gaussian integral well defined, but also the right-hand side of (20) is now real. Further, we may again use $X_\tau = X_0\xi_\tau$ to extract the sole dependence of $\mathcal{P}$ on x, given that neither $\mathcal{M}'$ nor ξ are functions of x. As a result, the integration over μ leads to a factor proportional to

$$1/\sqrt{\mathcal{P}} \propto X_0^{(1-N)}. \tag{36}$$

Third, for Schrödinger problems in one dimension, the energy levels are non-degenerate, *unless* its potential is a constant. The implication for us is that, except for the $N = 2$ case, the spectrum of $\mathcal{M}$ is non-degenerate. Further, since our interval is finite, the spectrum is discrete, with eigenvalues increasing monotonically with the number of nodes in the eigenfunction. Now, the constraint $\delta(\int_0^1 \mathrm{d}\tau\ \chi_\tau)$ must clearly be satisfied by both $\hat{\chi}$ and the eigenfunctions. Therefore, the latter must have an even number ($2n \geqslant 2$) of nodes. We will show that the zero mode has the lowest allowed n. Since it is not degenerate, the rest of the spectrum of $\mathcal{M}$, i.e. the spectrum of $\mathcal{M}'$, must be positive.

Now, let us provide some details of the zero mode. Its origin is the translational invariance in our problem. To identify it explicitly, differentiate (22) once and find

$$\mathcal{M}\frac{\mathrm{d}X_\tau}{\mathrm{d}\tau} = 0. \tag{37}$$

Thus, $\dot{X}_\tau$ is an eigenfunction of $\mathcal{M}$ with zero eigenvalue, so that it is clearly the zero mode. Next, to show that it is also absent from the first term in (33), we note that $\mathrm{sgn}(X_\tau)|X_\tau|^{(N-1)}$ is actually $\mathrm{d}V(X)/\mathrm{d}X$. Therefore, if $\hat{\chi}_\tau$ is $\mathrm{d}X_\tau/\mathrm{d}\tau$, then the integrand is a perfect derivative of a periodic function and this term vanishes. The conclusion is that the zero mode is completely absent from the quadratic from (33). Thus, for the Gaussian approximation to be valid this mode has to be treated separately. The standard technique to deal with this situation is the method of collective coordinates [4]. The symmetry here is translational invariance of (21) and (22), which implies that the particle (in the mechanical analogy) can be at the point X_0 at any other time $\tau_0 \in [0, 1]$. Thus, there is a one-parameter family of solutions, labelled by τ_0, all of which satisfy the differential equation (22) and the constraint $\int_0^1 \mathrm{d}\tau\, \chi_\tau = 0$. Indeed, the zero mode $\dot{X}_\tau$ can be recognized as the difference between X_τ and $X_{\tau-\tau_0}$, to lowest order in τ_0. Using this parameter instead of the amplitude of $\dot{X}_\tau$ in the normal mode expansion of $\hat{\chi}$ enables us to sum over this component of $\hat{\chi}$, even though it is entirely absent from the weights.

Having made these remarks, we may now evaluate the Gaussian integral over $\hat{\chi}$. In principle, we would expand $\hat{\chi}$ in terms of all the eigenfunctions of $\mathcal{M}$, except the zero mode (associated with the index $n = 0$ below). Therefore, $\hat{\chi}_\tau = \sum_{n\neq 0} a_n \hat{\chi}_\tau^{(n)}$. In the collective coordinates method [4], the integral over the functions $\hat{\chi}$ is replaced by integrals over τ_0 (which 'replaces' a_0) and $\{a_n | n \neq 0\}$. The Jacobian of this transformation is simply $\langle \dot{X} | \dot{X} \rangle^{1/2}$, which is proportional to X_0 and carries the sole x dependence here. Now, the integration τ_0 is trivially unity. Meanwhile the integration over the rest of the modes ($\{a_n | n \neq 0\}$) has already been discussed, in connection with the first remark above. No x dependence appears here. Finally, the integration over $\hat{\mu}$ leads to (36). Summarizing, the Gaussian integrals yield a prefactor for (31) proportional to $X_0^{(2-N)}$. Using (29), we have our final result:

$$\ln \Phi_N(x) \sim -\mathcal{S}_N x^{2/N} - \frac{N-2}{N}\ln(x) + \mathrm{O}(1) \qquad (x \gg 1) \tag{38}$$

where, explicitly,

$$\mathcal{S}_N = \frac{4\pi}{3N(N+2)}\left(\frac{\Gamma(1/N)}{\Gamma((N+2)/2N)}\right)^2 \left\{\frac{(N+2)\Gamma((N+1)/2)}{2\sqrt{\pi}}\right\}^{2/N}. \tag{39}$$

Although there are some technical limitations to our derivation for the $N = 2$ case (e.g. equation (26)), they can be circumvented, so that this result is in fact valid for $N \geqslant 2$.

4. The scaling function for small x

To arrive at Φ_N in the other limit, $x \ll 1$, we will use a very different approach, which we will first sketch briefly. As in the $x \gg 1$ case, we are only able to compute the two leading terms in $\ln \Phi_N$, so that all proportionality constants will be dropped. Unlike in the previous section, we will first compute the asymptotic behaviour of $G_N(\lambda; T)$ for large λ and then rely on a saddle point method to find Φ_N. Our starting point is the unscaled version of $G_N(\lambda; T)$ given by (9). So first consider the numerator of the right-hand side, but without the constraint $\int_0^T \mathrm{d}t\, \psi_t = 0$, and define

$$\alpha \equiv \lambda/T. \tag{40}$$

Then,

$$\int \mathcal{D}[\psi] \exp - \int_0^T \mathrm{d}t\,[\tfrac{1}{2}\dot{\psi}_t^2 + \alpha|\psi_t|^N] \propto \sum_n \mathrm{e}^{-E_n^{(N)}T} \tag{41}$$

where $E_n^{(N)}$ is the $(n+1)$th eigenvalue of the evolution operator which consists of the quantum-mechanical Hamiltonian for a particle moving in the potential $\alpha|\psi|^N$. As $T \to \infty$, the ground state dominates, leading to the Feynman–Kac result [5] that the right-hand side of (41) can be replaced by $\mathrm{e}^{-E_0^{(N)}T}$, with only an exponentially small error. Now, the dependence of $E_0^{(N)}$ on α can be extracted through dimensional analysis alone, since α has dimensions of inverse time to the power $\frac{1}{2}(N+2)$, while $E_0^{(N)}$ itself has dimensions of inverse time. Thus, we have $E_0^{(N)} \propto \alpha^{2/(N+2)}$ and arrive at the the leading behaviour of the path integral in (41), for $T \gg \alpha^{-2/(N+2)}$.

Having outlined the essential idea, let us now be more systematic. First, to implement the constraint $\int_0^T \mathrm{d}t\,\psi_t = 0$, we write the usual integral representation for a delta function. Next, instead of normalizing the path integral as in (9), we use $G_2(\lambda; T)$ as the denominator. This avoids the complications from the $\alpha = 0$ system, which lacks a discrete spectrum for a simple application of the Feynman–Kac result. On the other hand, $G_2(\lambda; T)$ is known explicitly (see (4)), so that the ratio G_N/G_2 is essentially our goal. Thus, we consider, from (9),

$$\frac{G_N(\lambda;T)}{G_2(\lambda;T)} = \frac{\int_{-\infty}^{\infty}\mathrm{d}\omega \int \mathcal{D}[\psi]\exp - \int_0^T \mathrm{d}t\,[\frac{1}{2}\dot{\psi}_t^2 + \alpha|\psi_t|^N + \mathrm{i}\omega\psi_t]}{\int_{-\infty}^{\infty}\mathrm{d}\omega \int \mathcal{D}[\psi]\exp - \int_0^T \mathrm{d}t\,[\frac{1}{2}\dot{\psi}_t^2 + \alpha\psi_t^2 + \mathrm{i}\omega\psi_t]}. \tag{42}$$

Of course, the presence of $\mathrm{i}\omega\psi$ is rather unusual. However, as we will see, its sole effect is to give a non-trivial prefactor to G in this limit. We first focus on the denominator, which consists of simple Gaussian integrals. Without the constraint, the $\mathrm{i}\omega\psi$ term is absent, so that we may use results from the quantum mechanics of a simple harmonic oscillator to obtain $G_2 \sim \mathrm{e}^{-\sqrt{\alpha/2}T}$. Comparing with (4), we see that this is indeed the leading behaviour. The next leading term, contained in the prefactor, would arise from the constraint. In this case it is easily dealt with by defining $\psi_t' = \psi_t + \mathrm{i}\omega/2\alpha$ and factorizing the integrand. The ψ' integral gives us the previous asymptotic form, while integration over ω produces the desired prefactor: $\sqrt{\alpha/T}$. For the $N \neq 2$ case, there is no similar luxury of factorization, but, by reinterpreting the above steps, i.e. computing the functional integral with the $\mathrm{i}\omega\psi$ term as finding an ω-dependent ground-state energy, $E_0^{(2)}(\alpha,\omega)$, a way forward can be seen. Thus, we write

$$\int \mathcal{D}[\psi] \exp - \int_0^T \mathrm{d}t\,[\tfrac{1}{2}\dot{\psi}_t^2 + \alpha\psi_t^2 + \mathrm{i}\omega\psi_t] \propto \sum_n \mathrm{e}^{-E_n^{(2)}(\alpha,\omega)T} \approx \mathrm{e}^{-E_0^{(2)}(\alpha,\omega)T} \tag{43}$$

with

$$E_0^{(2)}(\alpha,\omega) = E_0^{(2)}(\alpha,0) + \omega^2/4\alpha. \tag{44}$$

It is now clear that the integral over ω gives the prefactor. Another route to the understanding of (44) is to consider adding a real, linear potential, $h\psi$, first. Clearly, the ground-state energy is well defined, allowing us to arrive at the above result by analytic continuation to pure imaginary h.

Generalizing this approach to the $N \neq 2$ case, we write the numerator of (42) as

$$\int_{-\infty}^{\infty} \mathrm{d}\omega \sum_n \mathrm{e}^{-E_n^{(N)}(\alpha,\omega)T} \approx \int_{-\infty}^{\infty} \mathrm{d}\omega\, \mathrm{e}^{-E_0^{(N)}(\alpha,\omega)T}. \tag{45}$$

Unlike the $N = 2$ result (44), $E_0^{(N)}(\alpha, \omega)$ does not have a dependence on ω which terminates at second order. It is clear, however, by expanding the exponentials in the analogue of (43) for general N, that it has the general form

$$E_0^{(N)}(\alpha, \omega) = \varepsilon_0^{(N)}(\alpha) + \omega^2 \varepsilon_1^{(N)}(\alpha) + \omega^4 \varepsilon_2^{(N)}(\alpha) + \cdots. \tag{46}$$

So since ω has dimensions of time to the power $-3/2$, $\varepsilon_m^{(N)}(\alpha)$ has dimensions of time to the power $(3m - 1)$. Therefore,

$$\varepsilon_m^{(N)}(\alpha) = \varepsilon_m^{(N)}(1)\alpha^{-2(3m-1)/(N+2)}. \tag{47}$$

To see that only the first two terms in this expansion are relevant, recall that the integrand depends on $E_0^{(N)}(\alpha, \omega)T$ only. Changing the variable of integration from ω to $\omega' \equiv T^{1/2}\alpha^{-2/(N+2)}\omega$, so that the quadratic term in (46) is independent of α and T, we see that the $m \geqslant 2$ terms are of order $1/[\alpha^{2/(N+2)}T]^{m-1}$. Since this is proportional to $1/\mu^{2(m-1)/(N+2)}$ and we are interested in the $\mu \gg 1$ limit, we are justified in neglecting all terms beyond $m = 1$. Focusing on the first two terms in (46), we write

$$E_0^{(N)}(\alpha, \omega)T = \mathcal{E}\alpha^{2/(N+2)}T + \mathcal{E}'(\omega')^2 + \cdots. \tag{48}$$

Note that $\mathcal{E} \equiv \varepsilon_0^{(N)}(1)$ is just the zero point energy of a particle in the potential $|\zeta|^N$ and is, therefore, positive. Meanwhile, had ω been pure imaginary, $E_0^{(N)}$ would certainly have been lowered, implying that $\mathcal{E}'$ is also positive. We note, parenthetically, that it can be estimated in our approach, since it is just the correction to the ground-state energy in second-order perturbation theory. Having $\mathcal{E}' > 0$, the integral over ω' is well defined and provides a simple constant.

Inserting (48) into (45), we have

$$\int_{-\infty}^{\infty} \mathrm{d}\omega\, \mathrm{e}^{-E_0^{(N)}(\alpha,\omega)T} \propto \alpha^{2/(N+2)}T^{-1/2}\exp[-\mathcal{E}\alpha^{2/(N+2)}T]\{1 + \mathrm{O}(\alpha^{-2/(N+2)}\,T)\}. \tag{49}$$

Using this, together with a similar expression for the denominator,

$$\int_{-\infty}^{\infty} \mathrm{d}\omega\, \mathrm{e}^{-E_0^{(2)}(\alpha,\omega)T} \propto \alpha^{1/2}T^{-1/2}\exp[-\alpha^{1/2}T/\sqrt{2}] \tag{50}$$

we find

$$\frac{G_N(\lambda; T)}{G_2(\lambda; T)} \propto \frac{\lambda^{2/(N+2)}}{\lambda^{1/2}}\,\frac{T^{-2/(N+2)}}{T^{-1/2}}\,\frac{\exp[-\mathcal{E}\lambda^{2/(N+2)}T^{N/(N+2)}]}{\exp[-\lambda^{1/2}T^{1/2}/\sqrt{2}]}\{1 + \mathrm{O}(\lambda T^{N/2})^{-2/(N+2)}\} \tag{51}$$

where we replaced the α's by the original λ/T's. Finally, using the exact expression (4) for G_2, we arrive at

$$G_N(\lambda; T) \propto \lambda^{2/(N+2)}T^{N/(N+2)}\exp[-\mathcal{E}\lambda^{2/(N+2)}T^{N/(N+2)}]\{1 + \mathrm{O}(\lambda T^{N/2})^{-2/(N+2)}\}. \tag{52}$$

Alternatively, we can write it in terms of the scaling form $g_N(\mu)$ introduced in section 2:

$$g_N(\mu) \propto \mu^{2/(N+2)}\exp[-\mathcal{E}(\mu/N)^{2/(N+2)}]\{1 + \mathrm{O}(\mu^{-2/(N+2)})\}. \tag{53}$$

This is the form which G_N or g_N takes in the $\mu \gg 1$ limit. It should be contrasted with the approach used in the last section, which was justified in the regime where $\mu \ll 1$. Thus the results of the last section are complementary to those derived here.

All that remains to find $\Phi_N(x)$ is to take the inverse Laplace transform of (53):

$$\Phi_N(x) \propto \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}\mu\, \mu^{2/(N+2)}\exp[-\mathcal{E}(\mu/N)^{2/(N+2)} + \mu x|g_N'(0)|]. \tag{54}$$

Defining $\nu = x^{(N+2)/N}\mu$ gives, for the exponent in (54),

$$x^{-2/N}\{-\mathcal{E}(\nu/N)^{2/(N+2)} + \nu|g_N'(0)|\} \tag{55}$$

so that the integral may be evaluated by steepest descent if $x \ll 1$. Since $\mathcal{E} > 0$, there is an extremum at a real positive value of ν, which is of order unity. Thus, the corresponding value of μ is of order $x^{-(N+2)/N} \gg 1$, justifying all the approximations we made above, based on $\mu \to \infty$. Carrying through the integration gives the asymptotic behaviour:

$$\ln \Phi_N(x) \sim -\mathcal{K}_N x^{-2/N} - \frac{N+3}{N} \ln(x) + \mathrm{O}(1) \qquad (x \ll 1) \tag{56}$$

where $\mathcal{K}_N$ is a positive constant which depends on some explicitly known functions of N and on $\mathcal{E}$, the ground-state energy of a quantum-mechanical particle of unit mass in a potential $V(\zeta) = |\zeta|^N$. Unfortunately, the last item is not known for general N, so that we must be content with just an implicit expression. Nevertheless, conjecturing that it is monotonic in N, we can place a bound on it, $\pi^2/8$, which is the result for $N \to \infty$, where the particle is confined to $|\zeta| \leqslant 1$ by an infinite square well. Since $\mathcal{E}$ is $1/\sqrt{2}$ for $N = 2$, it is fairly well bounded numerically even though its precise value is unknown. Finally, we note that (56) reduces to the known result given in [1] for the $N = 2$ case.

5. Conclusions

To summarize, we have generalized the study of width distributions of a Gaussian path (random walk) to include moments higher than two. Unlike the $N = 2$ case, we are unable to find closed form expressions for the Laplace transform of these distributions. Nevertheless, we can compute their asymptotic behaviour, for both large and small arguments: equations (38) and (56). Here, we conclude with a few remarks.

First, note that, though the *averages* of the even $N > 2$ moments are trivially related to $\langle w^2 \rangle$, their *distributions* are not so simple. In particular, we believe that there is no way to obtain the general Φ_N or G_N from the $N = 2$ result. They contain different information about the paths. For example, the following two paths lead to the same w^2 but distinct $w^{(N)}$: (i) $\psi_t = 1$ for $0 < t < T/2$, -1 for $T/2 < t < T$ and 0 elsewhere; compared with (ii) $\psi_t = 2$ for $0 < t < T/8$, -2 for $7T/8 < t < T$ and 0 elsewhere. Thus, only paths with short excursions to large ψ's, which are presumably rarer, contribute to the large x tail of Φ_N with large N. Presumably, this is reflected by equation (31).

Second, we have seen, from Feynman's path-integral formulation of quantum mechanics, that the Laplace transform of the distributions for the Nth absolute central moments of random walks are intimately related to the propagation kernels of a quantum-mechanical particle confined to a potential of the form $V(\zeta) = |\zeta|^N$. As a result, the behaviour of $\Phi_N(x)$ in the large x limit should be controlled by the properties of $G_N(\lambda)$ for small λ. We note that the paths which dominate G_N here are the ones near the 'classical path' (22). Meanwhile, for the opposite limit, small x and large λ, the dominant contribution to G_N comes from the ground state, which is, in a sense, 'the furthest from classical'. It is interesting that these opposing aspects of the quantum mechanical problem find their way into the opposite ends of the asymptotics.

Third, let us recall that these distribution functions are supposedly universal, in the language of a renormalization group. In particular, our results represent new features of the class of simple random walks, i.e. a massless Gaussian fixed point (also referred to in the dynamics of interfaces as the Edwards–Wilkinson [6] universality class). As universal quantities, the distribution functions should capture the large-scale properties of a one-dimensional object stabilized by non-zero tension, independent of the microscopics of the system. Thus, we may expect the same behaviour from an interface between different phases, a polymer in $d = 2$, or a simple model of random walk. The last of these is particularly easy to study, using Monte Carlo simulations. In this connection, we should caution the reader on the $N \to \infty$ limit. Simulations necessarily deal with systems with finite L. To observe universal properties, we must let L approach infinity. However, this does not commute with $N \to \infty$, so the results above should be used with some care if large values of N were to be used in the investigation.

Finally, it is natural to speculate on further generalizations of this study. Obvious candidates are interfaces in higher dimensions [7] and those controlled mainly by curvature terms [8]. Both will involve single component fields. On the other hand, we could investigate random walks imbedded in higher dimensions, such as physical polymers in three-dimensional solutions, where the periodic boundary condition would correspond to ring polymers. In this case, we would need multi-component fields and an appropriate generalization of the concept of 'width'. In the simplest scenario, $P(w^2)$ will be nothing more than products of the distribution in [1]. However, it is clear that P_N will be much more interesting! While the formulation of any of these problems is trivial, we anticipate that the analysis itself will be be rather difficult.

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